Division of space by Voronoi graphs, percolation within percolation and application to the models of porous membranes

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No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university, or other institute of learning.

K N Tiyapan

To my father and my mother Niwat and Somjit Tiyapan (*Nivat, Somcitṫa Tịyȧbanḋha)*,
my chemistry and sword teacher Siddhiponr Songsataya (*Sidhibaur Drongsatya*), and
my dad and mum Allister and Patricia Johnstone.
To Emeritus Professor David John Bell and
Emeritus Professor Graham Arthur Davies,
both of whom had started this project,
one of whom had powered it.
To these people, then, is my dedication of this thesis,
but the thesis may dedicate itself to whomever it wishes.

Abstract

Statistics for Voronoi Tessellations are obtained from numerical simulation. These are, for example, the number of edges, faces, and vertices per cell, and the cell's surface area and volume. Statistics are obtained for the 2-d section of 3-d Voronoi networks and for compressed VT. For networks of from two to ten dimensions are given the number of vertices per cell and the ratio between the volume and the hypercubic volume. The similarity between a percolation curve and an s-curve is discussed. Percolation is applied to the study of traffic congestion. In filtration problems sometimes tiny particles come together to form clusters which block the pores. This is due to the Van der Waals force which acts to pull them together and bind them. It is proposed here that such phenomena may be represented as a percolation within percolation, or more precisely a *continuum* percolation within a *network* percolation. Here channelling occurred in filters causes the concentration of the suspension to fluctuate. Pores whose internal concentration exceeds the continuum percolation threshold become blocked. This blockage may not be wholly random but depend on the quantum-mechanical wavefunction of the particles when their wave-particle duality are taken into account, as described by the de Broglie formula $\lambda = h/p$. These blocked pores then act to percolate the filter as a network percolation problem. Algorithms and programs for doing the simulation and compiling the thesis are given, and so are translation of three seminal papers by Voronoi (1908 and 1909) and another one by Dirichlet (1850).

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Above all I thank God for letting me be and for everything.

No writings can be too long
nor tunes futile in any song
where to someone thanks we say
and in our heart for them we pray.

Yet at times we forget
and amidst our busy tasks let
all the pains in and here reside.
We could have let God us abide.

O, gentle reader, never judge!
We last no longer while we grudge!
Search again, and you shall find so much
there is nothing meaningless as such.

Kit Tiyapan
Manchester, 2003

The above poem was written on 24^{th} September 2003 . By the end of the month I have already given a copy of it to many people, and many more in the form of emails to friends in the UK, the US, Japan and Thailand. This is how I come to realise that I have missed out quite a few of the names of those people who have helped. If I had to rid my writing of all the stuffs it is made of, I would never have even dreamt about ridding it of each and every one of all the thanks I have put therein. I thank Peter Fawcett for having had decided to supervise my projects on languages though in the end I had to do them all by myself and on my own. I thank Graham Davies again for the full two years of financial support, the inspiration, compassion and encouragement received. I thank Mona Baker for having given me an office and access to a unix machine at the translation group. Though I had to return them all to her for my works for Graham, it had been a wonderful experience for me to have done some work for her translational corpus.

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Notation

$0(n)$. a zero vector of n dimensions.

$0(n, n)$. an $n \times n$ zero-matrix.

$1(n)$. a one vector of n dimensions.

$1(n, n)$. an $n \times n$ one-matrix.

a. surface area of a cell.

a^{ij}. cofactor of a_{ij}, $a^{ij} = (-1)^{i+j} \det(a_i, b_i)$.

$\aleph$. neighbour.

α. surface area per unit volume.

β. area of face.

c. cell; concentration.

d. length.

D. diffusion coefficient.

δ. thicness of the boundary layer.

ϵ. porosity.

ϵ_0. permittivity of free space, $\epsilon_0 = 8.854 \times 10^{-12}$ Fm^{-1}.

η. viscosity of liquid.

e. edge; the charge of an electron, $e = -q$. See q.

$E(x)$. expected number of x

f. form factor, f_c, of a cell, f_f, of a face; face, nf, n-face.

h. Planck's constant, 6.626×10^{-34} J · s.

he. he; he or she.

I(n). an $n \times n$ identity matrix.

I_r. integers in the interval r *eg*, $I_{[0,\infty)}$.

J. flux.

κ. Kurtosis.

$\mathcal{K}$. Kurtosis.

λ. wavelength. m.

μ. weight of particles in the cake per unit of filter surface; fluid viscosity; magnetic moment.

$\mu(\cdot)$. mean; also μ_g the geometrical mean, μ_h the harmonic mean.

$m^n(\cdot)$. the n^{th}-moment of.

mad$(\cdot)$. mean absolute deviation.

med$(\cdot)$. median.

$\mathcal{M}^n(\cdot)$. the n^{th}-moment of.

N. original total number, for example N$_c$ is the total number of cells, and N$_v$ of vertices, created which may include those that are out of bound.

n_x. number of x. n_c, number of cells. n_e, number of edges. n_v, number of vertices.

n^x_y. number of x of a y. n^e_c, number of edges of a cell. n^f_c, number of faces of a cell. n^v_c, number of vertices of a cell.

$n^x_{y,z}$. number of x per y of z. $n^e_{f,c}$, number of edges per face of a cell.

ω. angular velocity.

p. probability; (with subscript) percolation probability. p_c, p_b, p_v, p_e, percolation probability of cells, bonds, vertices, and edges respectively. alternatively, p^c_c, p^b_c, p^v_c, p^e_c,.

$p(\cdot)$. probability density function, probability distribution.

P. pressure.

q. the charge of a proton, 1.6×10^{-19} C; feed rate.

Q. flow rate, $Q = \mathrm{d}V/\mathrm{d}t$.

r. radius. r_v, r_e, r_f, radius to a vertex, edge, or face, respectively.

ρ. density; number density.

r. specific resistance.

§. (s) section or chapter.

s. perimeter, for example s_c the perimeter of cell; cell perimeter.

σ. spin variable.

θ. time.

v. velocity.

vertice. any vertex.

V. volume.

W. fluid velocity.

x. coordinates; coordination number.

$\bar{x}$. mean value of x, *cf* $E(x)$.

z. face perimeter.